THE LEXICOGRAPHIC DEGREE OF THE FIRST TWO-BRIDGE KNOTS

EXTENDED ABSTRACT

ERWAN BRUGALLÉ, PIERRE-VINCENT KOSELEFF, DANIEL PECKER

ABSTRACT. We study the degree of polynomial representations of knots. We give here the lexicographic degree of all knots with eight or fewer crossings. The proof uses the braid theoretical method developed by Orevkov to study real plane curves, isotopies on trigonal curves and explicit parametrizations obtained by perturbing a triple point.

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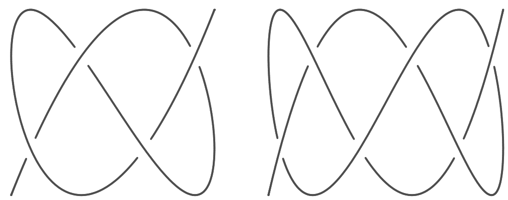
Erwan BRUGALLÉ
École Polytechnique, Centre Mathématiques Laurent Schwartz, France
e-mail: `erwan.brugalle@math.cnrs.fr`

Pierre-Vincent KOSELEFF
Université Pierre et Marie Curie (UPMC Sorbonne Universités),
Institut de Mathématiques de Jussieu (IMJ-PRG) & Inria-Rocquencourt
e-mail: `koseleff@math.jussieu.fr`

Daniel PECKER
Université Pierre et Marie Curie (UPMC Sorbonne Universités),
Institut de Mathématiques de Jussieu (IMJ-PRG),
e-mail: `pecker@math.jussieu.fr`

1. Introduction

Vassiliev proved that every knot in $\mathbf{S}^3$ can be represented as the closure of the image of a polynomial embedding $\mathbf{R} \to \mathbf{R}^3 \subset \mathbf{S}_3$, see [Sh, Va].



deg $4_1 = (3, 5, 7)$ deg $5_1 = (3, 7, 8)$

Figure 1. *xy-diagrams* of the figure-eight knot 4_1 and the torus knot 5_1

The *multidegree* of a polynomial map $\gamma : \mathbf{R} \to \mathbf{R}^n, t \mapsto (P_i(t))$ is the n-tuple $(\deg(P_i))$. The *lexicographic degree* of a knot K is the minimal multidegree, for the lexicographic order, of a polynomial knot whose closure in $\mathbf{S}^3$ is isotopic to K. The unknot has lexicographic degree $(-\infty, -\infty, 1)$, and one sees easily that the lexicographic degree of any other knot is (a, b, c) with $3 \le a < b < c$. Given a knot, it is in general a difficult problem to determine its lexicographic degree. In particular, the corresponding diagram might not have the minimal number of crossings.

The *two-bridge* knots are precisely those with lexicographic degree $(3, b, c)$, see [Cr, KP3, BKP1]. It means that they have a xy-projection which is a trigonal curve.

The aim of this paper is to give the lexicographic degree of the first 26 two-bridge knots with eight or fewer crossings. The constructions use tools previously developed: Orevkov's braid theoretical approach to study real plane curves and the use of *real pseudoholomorphic curves* ([BKP2]), the *slide isotopies* on trigonal diagrams, namely those that never increase the number of crossings ([BKP1]) and explicit parametrizations of knots ([KP1, KP2, KP3, KPR]).

In addition, we introduce a new reduction R (see Def. 3.3) on trigonal plane curves. In many cases, the use of this reduction allows to subtract three to the number of crossings of the diagram and to the degree b, thanks to a result on real pseudoholomorphic curves (Th. 3.4), deduced from [Or2]. On the other hand we show that we can explicitly give a polynomial curve of bidegree $(3, d + 3)$ from a polynomial curve of degree $(3, d)$ by adding a triple point and perturbing the singularity (Prop. 3.5). We also deduce a sharp upper bound for c (Prop. 4.3).

The extended abstract is organised as follows. In Section 2 we recall some properties of two-bridge knots and their trigonal diagrams. In Section 3, we consider plane trigonal curves and we describe how we obtain a lower bound for the lexicographic degree of pseudoholomorphic curves and therefore for polynomial embeddings. In Section 4, we explain how we obtain our bounds and how we obtain the lexicographic degrees of our 26 first two-bridge knots with eight or fewer crossings.

2. Trigonal diagrams of two-bridge knots

A two-bridge knot admits a diagram in *Conway's open form* (or trigonal form). This diagram, denoted by $D(m_1, m_2, \ldots, m_k)$ where $m_i \in \mathbf{Z}$, is explained by the following picture (see [Co], [Mu, p. 187]). The number of twists is denoted by the

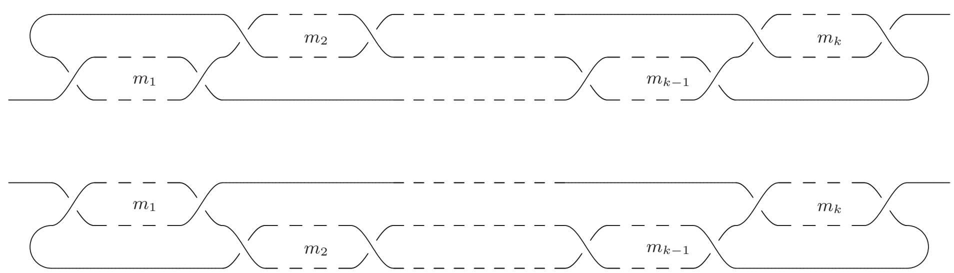


Figure 2. Conway's form for links

integer $|m_i|$, and the sign of m_i is defined as follows: if i is odd, then the right twist is positive, if i is even, then the right twist is negative. In Figure 3 are $D(0, -1, -3)$, $D(3, 0, -1, -2)$ as examples.

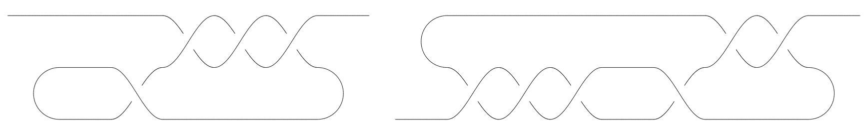


Figure 3. Examples of trigonal diagrams

These diagrams are also called 3-strand-braid representations, see [KL].

The two-bridge links are classified by their Schubert fractions

$$\frac{\alpha}{\beta} = m_1 + \cfrac{1}{m_2 + \cfrac{1}{\cdots + \cfrac{1}{m_k}}} = [m_1, \ldots, m_k], \quad \alpha \geq 0, \ (\alpha, \beta) = 1.$$

Given $[m_1, \ldots, m_k] = \dfrac{\alpha}{\beta}$ and $[m'_1, \ldots, m'_l] = \dfrac{\alpha'}{\beta'}$, the diagrams $D(m_1, m_2, \ldots, m_k)$ and $D(m'_1, m'_2, \ldots, m'_l)$ correspond to isotopic links if and only if $\alpha = \alpha'$ and $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$, see [Mu, Theorem 9.3.3]. The integer α is odd for a knot, and even for a two-component link.

Every positive fraction α/β admits a continued fraction expansion $[m_1, \ldots, m_k]$ where all the m_i are positive. Therefore every two-bridge knot K admits a diagram in *Conway's normal form*, that is an alternating diagram of the form $D(m_1, m_2, \ldots m_k)$, where the m_i are all positive or all negative. In this case we write $L = C(m_1, \ldots, m_k)$.

Definition 2.1. *We define the complexity of a trigonal diagram* $D(m_1, \ldots, m_k)$ *as* $c(D) = k + \sum |m_i|$.

The alternating diagram $D(m_1, \ldots, m_k)$ has the smallest complexity among the diagrams of $C(m_1, \ldots, m_k)$. It is classical that one can transform any trigonal diagram of a two-bridge knot into its Conway's normal form using the Lagrange isotopies, see [KL] or [Cr, p. 204]:

$$D(x, m, -n, -y) \to D(x, m - \varepsilon, \varepsilon, n - \varepsilon, y), \ \varepsilon = \pm 1, \tag{1}$$

where m, n are integers, and x, y are sequences of integers (possibly empty), see Figure 4. These isotopies twist a part of the diagram, and the number of crossings

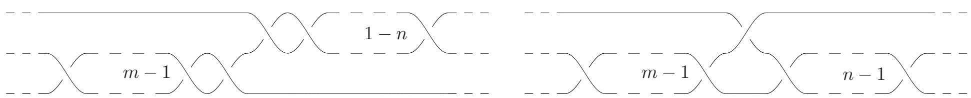


Figure 4. Lagrange isotopy: $D(x, m, -n, -y) \to D(x, m - 1, 1, n - 1, y)$

may increase in intermediate diagrams.

Definition 2.2. *We shall say that an isotopy of trigonal diagrams is a* slide *isotopy if the number of crossings never increases and if all the intermediate diagrams remain trigonal.*

Definition 2.3. *A trigonal diagram is called a* simple *diagram if it cannot be simplified into a diagram of lower complexity by using slide isotopies only.*

In [BKP1] we proved the following:

Theorem 2.4. *Let D be a trigonal Conway diagram of a two-bridge link. Then by slide isotopies, it is possible to transform D into a simple diagram $D(m_1, \ldots, m_k)$ such that for $i = 2, \ldots, k$, either $|m_i| \neq 1$, or $m_{i-1}\, m_i > 0$.*

3. Plane trigonal curves

Once we get a simple diagram $D(m_1, \ldots, m_k)$, we first examine its projection $|D|$. Note that $|D|$ does not depend on the signs of the m_i's.

Definition 3.1. *An isotopy on a curve $C \subset \mathbf{C}^2$ is called a $\mathcal{L}$-*isotopy *if it commutes with the projection $\pi : \mathbf{C}^2 \to \mathbf{C}$, $(x, y) \mapsto x$.*

We make use of the next result ([BKP2, Cor. 3.7]) deduced from [Or1]:

Theorem 3.2. *Let D_1 and D_2 be two trigonal knot diagrams such that D_2 is obtained from D_1 by a slide isotopy. If there exists a real nodal pseudoholomorphic curve $\gamma_1 : \mathbf{C} \to \mathbf{C}^2$ of bidegree $(3, b)$ such that $\gamma_1(\mathbf{R})$ is $\mathcal{L}$-isotopic to $|D_1|$, then there also exists a real nodal pseudoholomorphic curve $\gamma_2 : \mathbf{C} \to \mathbf{C}^2$ of bidegree $(3, b)$ such that $\gamma_2(\mathbf{R})$ is $\mathcal{L}$-isotopic to $|D_2|$.*

In [BKP2], we study the case of trigonal diagrams $C(m)$ and $C(m, n)$ and show that their lexicographic degrees are exactly $(3, \left[\frac{3N-1}{2}\right], \left[\frac{3N}{2}\right] + 1)$, where N is the crossing number of the knot.

When the trigonal plane diagram is $D(m_1, \ldots, m_k)$, $k \geq 3$, then we will try to use the reduction R that decreases the number of crossings.

Definition 3.3. *Let u, v be (possibly empty) sequences of nonnegative integers and m, n be nonnegative integers. The reduction R transforms the plane diagram $D(u, m + 1, 1, n + 1, v)$ into the plane diagram $D(u, m, n, v)$.*

The reduction R is simply obtained by projecting the Lagrange isotopy (Formula 1). A result of Orevkov ([Or2, Prop. 2.2]) implies the following useful proposition:

Theorem 3.4. *Let D_1 and D_2 be two plane trigonal diagrams such that D_2 is obtained from D_1 by a reduction R. Then there exists a real nodal pseudo-holomorphic curve $\gamma_1 : \mathbf{C} \to \mathbf{C}^2$ of bidegree $(3, d)$ such that $\gamma_1(\mathbf{R})$ is $\mathcal{L}$-isotopic to D_1 if and only if there exists a real nodal pseudo-holomorphic curve $\gamma_2 : \mathbf{C} \to \mathbf{C}^2$ of bidegree $(3, d-3)$ such that $\gamma_2(\mathbf{R})$ is $\mathcal{L}$-isotopic to D_2.*

Furthermore we obtain polynomial curves with the following:

Proposition 3.5. *Let D_1 and D_2 be two plane trigonal diagrams such that D_2 is obtained from D_1 by a reduction R. Suppose that there exists a trigonal polynomial curve of degree $(3, d-3)$ with diagram D_2. Then there exists a trigonal polynomial curve of degree $(3, d)$ that is $\mathcal{L}$-isotopic to D_1.*

Proof. Let us start with a polynomial curve $\mathcal{C} : x = P_3(t),\ y = Q_d(t)$ that is $\mathcal{L}$-isotopic to the plane diagram $D(u, m, n, v)$, where u, v are (possibly empty) sequences of nonnegative integers and m, n are nonnegative integers. By translation on x, we can suppose that $[x = 0]$ separates the m crossings from the n crossings. We can also suppose that $[x = 0]$ meets $\mathcal{C}$ in three points with nonzero y-coordinates. The curve (x, xy) will have the same double points as $\mathcal{C}$ and an additional triple point at $x = y = 0$. We claim that for ε small enough the curve $(P_3(t+\varepsilon), P_3(t) \cdot Q_d(t))$ will be $\mathcal{L}$-isotopic to either $D(u, m+1, 1, n+1, v)$ or $D(u, m, 1, 1, 1, n, v)$, depending on the sign of ε. □

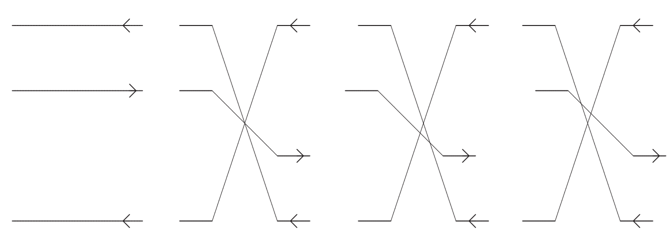


FIGURE 5. Perturbation of an added triple point

4. Results

4.1. **Upper bound with Chebyshev diagrams.** Let T_n be the classical Chebyshev polynomial $T_n(\cos t) = \cos nt$. It is known ([KP2]) that every knot admits a *Chebyshev diagram*, that is to say, there exist positive coprime integers a, b and a polynomial C of degree c such that the polynomial embedding (T_a, T_b, C) is the knot K.

Definition 4.1. *Let $\dfrac{\alpha}{\beta} = [a_1, \ldots, a_m]$, $a_i \in \mathbf{Z}^*$. We say that i is an islet in $[a_1, \ldots, a_m]$ if $a_i = \pm 1$ and $a_i a_{i+1} < 0$, $a_{i-1} a_i < 0$.*

Let K be a two-bridge knot with crossing number N. In [KP3, Sect. 3], we show that K has a diagram $D(\varepsilon_1, \ldots, \varepsilon_m)$, $\varepsilon_i = \pm 1$, of minimal length $m_C(K) \le \frac{3}{2}N - 2$. This means that K admits the Chebyshev diagram $C(3, b)$ where $b = m_C(K) + 1$.

It is also shown in [KP3] that K admits a polynomial parametrization (T_3, T_b, C) where $\deg C + b = 3N$. This gives a first upper bound $\deg_C(K)$ for the lexicographic degree of K.

4.2. **Lower bound.** We showed in [BKP2] that the lexicographic degree of a knot K is obtained by only considering simple diagrams of K.

It is shown in [BKP1] that two-bridge torus knots $C(m)$ and generalized twist knots $C(m, n)$ admit only one simple diagram. For this diagram we have an explicit polynomial embedding of degree $(3, \left[\frac{3N-1}{2}\right], \left[\frac{3N}{2}\right] + 1)$.

For two-bridge knots $C(m_1, \ldots, m_k)$, $k \geq 3$, we compute all simple diagrams with $m_C(K)$ crossings or fewer because the number of crossings of a plane curve of bidegree $(3, b)$ is bounded by $\frac{1}{2}(3-1)(b-1) = b-1$.

A simple diagram of K is obtained as a sequence $(m_1, \ldots, m_k) \in \mathbf{Z}^{*m}$ with no islet, such that $[m_1, \ldots, m_k]$ is a Schubert fraction of K. The set of such sequences is finite (see also Prop. 4.3).

Once we get a simple diagram, we reduce it by using the R transformation. We thus deduce lower bounds for b.

4.3. **Upper bound for the height.** Suppose now that we get a simple diagram $D(m_1, \ldots, m_k)$ (with no islet) of K, and that there exists a plane curve of bidegree $(3, b)$ that is $\mathcal{L}$-isotopic to D, then we use the following properties:

Definition 4.2. *Let $D(K)$ be the diagram of a knot K having crossing points corresponding to the parameters $t_1, \ldots, t_{2m}$. The* Gauss sequence *of D is defined by $g_k = 1$ if t_k corresponds to an overpass and $g_k = -1$ if the t_k is an underpass.*

Proposition 4.3. *Let $D(K)$ be the a diagram $D(m_1, \ldots, m_k)$ with no islet. Let $s = \#\{i; m_{i-1}m_i < 0\}$ be the number of sign changes in the sequence $(m_1, \ldots, m_k)$. Then*

(1) *The crossing number of K is $N = \sum_{i=1}^{k} |m_i| - s$.*
(2) *The number c of sign changes in the Gauss sequence of D satisfies*

$$c = 2N + s - 1 = 2\sum_{i=1}^{k} |m_i| - 3s - 1.$$

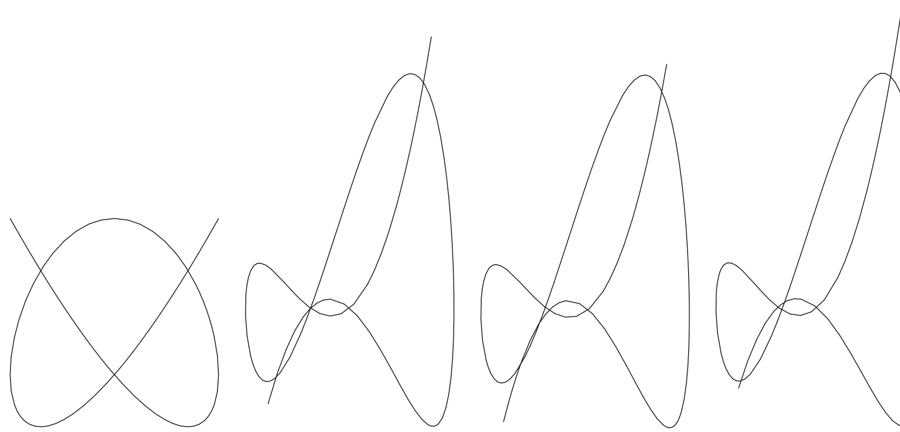
Proof. *The proof is analogous to the proof of* [KP3, Prop. 2.5 & Th. 5.2]. □

This gives an upper bound for the height function, by considering a polynomial with c prescribed sign changes.

Suppose now that $D(m_1, \ldots, m_k)$ is an alternating diagram of K. The number of sign changes in D is exactly $2N - 1$ and we proved in [BKP2, Th. 4.3] that if (P_3, P_{N+1}, P_c) is a polynomial curve of degree $(3, N+1, c)$ that is isotopic to D then $c \geq 2N + 1$.

4.4. **Adding three crossings.** We use Proposition 3.5 and show here how we obtain the knots $6_2 = C(2, 1, 3)$ and $6_3 = C(2, 1, 1, 2)$ from the plane diagram $D(3)$.

We start with a polynomial parametrization of the trefoil $D(1, 1, 1) \sim D(3)$. It is $(T_3(t), T_4(t))$. We choose to add a triple point in $(-3/4, 0)$ by considering the curve $x = T_3(t), y = Q_7(t)$ where $Q_7(t) = (T_3(t) + 3/4) \cdot (T_4(t) + 1)$. The curve in



(T_3, T_4) $D(3)$ — (T_3, Q_7) — $(T_3, Q_7(t+\varepsilon))$ $D(2,1,3)$ — $(T_3, Q_7(t-\varepsilon))$ $D(2,1,1,2)$

FIGURE 6. Adding three crossings to the trefoil

$(P_3(t), Q_7(t+\varepsilon))$ is $\mathcal{L}$-isotopic to $D(2,1,3)$ for $\varepsilon > 0$ small enough and is $\mathcal{L}$-isotopic to $D(2,1,1,2)$ for $\varepsilon < 0$.

We obtain the other constructions only using

- The degree of $D(0,1,1,0)$ is $(3,5)$. An explicit parametrization is given for example by $(t^3 - 3t, t^5 - 4t^3 + 4t)$ (see Figure 7). Suppose that there exists a polynomial embedding of degree $(3,4)$. A line joining the two crossings of the curve will cross the diagram at a fifth point, which is impossible.
- The degree of $D(0,2)$ or $D(2,0)$ is $(3,4)$. An explicit parametrization is given for example by $(t^3 - 3t, t^4 - 2t^2 - 2t - 2)$. (see Figure 7).
- The diagram $D(2,2) \sim D(1,1,1,1)$ is obtained for (T_3, T_5).
- The diagrams $D(1)$ is obtained for (T_3, T_2); more generally, the diagrams $D(2n+1)$ are obtained for (T_3, P_{3n+1}), where $\deg P_{3n+1} = 3n+1$, see [KP1].
- The degree of $D(0,1,3)$ is at least $(3,7)$.

$D(0,1,1,0)$ — $D(0,2) \sim D(0,1,1)$

FIGURE 7

Table. In the table, we list all two-bridge knots with their Schubert fraction. On the third column we give the degree $\deg_C(K)$ we obtain for Chebyshev diagrams, on the fourth column we list all simple diagrams with $m_C(K)$ or fewer crossings. On

the fifth column is the degree obtained by using the reduction R. The lexicographic degree is on the last column. It is starred when it is better than $\deg_C$.

K	α/β	$\deg_C$	Simple Diagrams	Degree	Bound	Lex. Degree
3_1	3	$(3,4,5)$	$D(3)$	$\deg D(3)+0$	$b\geq 4$	$(3,4,5)$
4_1	$5/2$	$(3,5,7)$	$D(2,2)$	$\deg D(2,2)+0$	$b\geq 5$	$(3,5,7)$
5_1	5	$(3,7,8)$	$D(5)$	$\deg D(5)+0$	$b\geq 7$	$(3,7,8)$
5_2	$7/2$	$(3,7,8)$	$D(2,3)$	$\deg D(2,3)+0$	$b\geq 7$	$(3,7,8)$
6_1	$9/2$	$(3,8,10)$	$D(2,4)$	$\deg D(2,4)+0$	$b\geq 8$	$(3,8,10)$
6_2	$11/3$	$(3,8,10)$	$D(2,1,3)$	$\deg D(3)+3$	$b=7$	$**(3,7,11)$
			$D(3,-4)$	$\deg D(3,4)+0$	$b\geq 10$	
6_3	$13/5$	$(3,7,11)$	$D(2,1,1,2)$	$\deg D(3)+3$	$b=7$	$(3,7,11)$
7_1	7	$(3,10,11)$	$D(7)$	$\deg D(7)+0$	$b\geq 10$	$(3,10,11)$
7_2	$11/2$	$(3,10,11)$	$D(2,5)$	$\deg D(2,5)+0$	$b\geq 10$	$(3,10,11)$
7_3	$13/3$	$(3,10,11)$	$D(3,4)$	$\deg D(3,4)+0$	$b\geq 10$	$(3,10,11)$
7_4	$15/4$	$(3,10,11)$	$D(3,1,3)$	$\deg D(1)+6$	$b\geq 8$	$**(3,8,13)$
			$D(4,-4)$	$\deg D(4,4)+0$	$b\geq 11$	
7_5	$17/5$	$(3,10,11)$	$D(2,2,3)$	$\deg D(0,1,3)+3$	$b\geq 10$	$(3,10,11)$
			$D(3,-2,4)$	$\deg D(3,2,4)+0$	$b\geq 10$	
7_6	$19/7$	$(3,10,11)$	$D(2,1,2,2)$	$\deg D(1)+6$	$b=8$	$**(3,8,13)$
			$D(2,3,-3)$	$\deg D(0,2,3)+3$	$b\geq 10$	
			$D(2,2,-2,3)$	$\deg D(0,1,2,3)+3$	$b\geq 10$	
7_7	$21/8$	$(3,8,13)$	$D(2,1,1,1,2)$	$\deg D(1)+6$	$b\geq 8$	$(3,8,13)$
8_1	$13/2$	$(3,11,13)$	$D(2,6)$	$\deg D(2,6)+0$	$b\geq 11$	$(3,11,13)$
8_2	$17/3$	$(3,11,13)$	$D(2,1,5)$	$\deg D(5)+3$	$b=10$	$**(3,10,c)$
			$D(3,-6)$	$\deg D(3,6)+0$	$b\geq 13$	
8_3	$17/4$	$(3,11,13)$	$D(4,4)$	$\deg D(4,4)+0$	$b\geq 11$	$(3,11,13)$
8_4	$19/4$	$(3,11,13)$	$D(3,1,4)$	$\deg D(0,2)+6$	$b=10$	$**(3,10,c)$
			$D(4,-5)$	$\deg D(4,5)+0$	$b\geq 13$	
8_6	$23/7$	$(3,11,13)$	$D(2,3,3)$	$\deg D(0,2,3)+3$	$b\geq 11$	$(3,11,13)$
8_7	$23/5$	$(3,10,14)$	$D(2,1,1,4)$	$\deg D(5)+3$	$b=10$	$(3,10,c)$
			$D(3,-2,-4)$	$\deg D(3,2,4)+0$	$b\geq 10$	
			$D(2,2,-5)$	$\deg D(0,1,5)+3$	$b\geq 10$	
8_8	$25/9$	$(3,10,14)$	$D(2,1,3,2)$	$\deg D(2,0)+6$	$b\geq 10$	$(3,10,c)$
			$D(2,4,-3)$	$\deg D(0,3,3)+3$	$b\geq 10$	
8_9	$25/7$	$(3,11,13)$	$D(3,1,1,3)$	$\deg D(5)+3$	$b=10$	$**(3,10,c)$
			$D(3,2,-4)$	$\deg D(3,2,4)+0$	$b\geq 10$	
8_{11}	$27/8$	$(3,11,13)$	$D(2,1,2,3)$	$\deg D(0,2)+6$	$b\geq 10$	$**(3,10,c)$
			$D(3,3,-3)$	$\deg D(3,3,3)+0$	$b\geq 10$	
			$D(3,-3,-3)$	$\deg D(3,3,3)+0$	$b\geq 10$	
			$D(2,2,-2,4)$	$\deg D(0,1,2,4)+3$	$b\geq 11$	
8_{12}	$29/12$	$(3,11,13)$	$D(2,2,2,2)$	$\deg D(0,1,1,0)+6$	$b\geq 11$	$(3,11,13)$
			$D(2,3,-2,3)$	$\deg D(0,2,2,3)+3$	$b\geq 11$	
8_{13}	$29/8$	$(3,10,14)$	$D(2,1,1,1,3)$	$\deg D(0,2)+6$	$b\geq 10$	$(3,10,c)$
			$D(3,1,2,-3)$	$\deg D(3)+6$	$b\geq 10$	
			$D(2,2,-2,-3)$	$\deg D(0,1,2,3)+3$	$b\geq 10$	
			$D(2,1,2,-4)$	$\deg D(0,3)+6$	$b\geq 10$	
			$D(3,-3,4)$	$\deg D(3,3,4)+0$	$b\geq 11$	

8_{14}	$31/12$	$(3, 11, 13)$	$D(2, 1, 1, 2, 2)$	$\deg D(2, 0) + 6$	$b \geq 10$	$^{**}(3, 10, c)$
			$D(2, 2, 2, -3)$	$\deg D(0, 1, 2, 3) + 3$	$b \geq 10$	
			$D(2, 2, -3, -2)$	$\deg D(0, 1, 2, 0) + 6$	$b \geq 10$	
			$D(2, 1, 2, -2, 3)$	$\deg D(0, 1, 3) + 6$	$b \geq 10$	

In most cases we obtain the lexicographic degree by considering the Chebyshev diagrams of K. We knew this fact for torus knots and generalized twist knots in [BKP2]. In the case of 6_2, 7_4, 7_6, the lexicographic degree is better.

In the case of 8_4, 8_7, 8_{11} and 8_{14} the lexicographic order is $(3, 10, c)$ and is better than the degree we obtain for Chebyshev diagrams. For these knots we do not know if $c = 11$ or $c = 14$.

References

[BKP1] E. Brugallé, P. -V. Koseleff, D. Pecker. *Untangling trigonal diagrams*, arXiv:1411.6367

[BKP2] E. Brugallé, P. -V. Koseleff, D. Pecker, *On the lexicographic degree of two-bridge knots*, arXiv:1411.5793

[Co] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), 329–358 Pergamon, Oxford (1970)

[Cr] P. R. Cromwell, *Knots and links*, Cambridge University Press, Cambridge, 2004.

[KL] L. H. Kauffman, S. Lambropoulou, *Hard unknots and collapsing tangles*, Introductory lectures on knot theory, Ser. Knots Everything, **46**, World Sci. Publ., Hackensack, 187-247, (2012)

[KP1] P. -V. Koseleff, D. Pecker, *A polynomial parametrization of torus knots*. Appl. Algebra Engrg. Comm. Comput. Vol **20 (5-6)** (2009), 361-377

[KP2] P. -V. Koseleff, D. Pecker, *Chebyshev knots*, Journal of Knot Theory and Its Ramifications, Vol 20, **4** (2011) 1-19

[KP3] P. -V. Koseleff, D. Pecker, *Chebyshev diagrams for two-bridge knots*, Geom. Dedicata 150 (2010), 405–425

[KPR] P. -V. Koseleff, D. Pecker, F. Rouillier, *The first rational Chebyshev knots*, Mega'09, J. Symbolic Comput. **45 (12)** (2010), 1341–1358.

[Mu] K. Murasugi, *Knot Theory and its Applications*, Boston, Birkhäuser, 341p., 1996.

[Or1] S. Yu. Orevkov. *Link theory and oval arrangements of real algebraic curves*, Topology, **38(4)**, 779–810, 1999.

[Or2] S. Yu. Orevkov. *Classification of flexible M-curves of degree 8 up to isotopy*, Geom. Funct. Anal., **12(4)**, 723–755, 2002.

[OS] S. Yu. Orevkov and E. I. Shustin. *Flexible, algebraically unrealizable curves: rehabilitation of Hilbert-Rohn-Gudkov approach*, J. Reine Angew. Math., **551**, 145–172, 2002.

[RS] A. Ranjan and R. Shukla, *On polynomial representation of torus knots,* Journal of knot theory and its ramifications, **5(2)** (1996) 279-294.

[Sh] A.R. Shastri, *Polynomial representation of knots*, Tôhoku Math. J. **44** (1992), 11-17.

[Va] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of singularities and its Applications, Advances Soviet Maths Vol 1 (1990)